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ON "FUNDAMENTAL SOLUTIONS FOR A FLUID-SATURATED POROUS SOLID" BY M. P. CLEARY

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Abstract—This note corrects expressions given by Cleary[1] for fluid mass source and dipole solutions in linear fluid infiltrated porous elastic solids. The corrected expressions reveal that the stress and displacement fields due to a suddenly applied point force comprise a time independent portion that is the classical elasticity solution based on the undrained (short-time) moduli, and a time dependent portion that is the solution for a continuous fluid mass dipole. As a corollary, the time dependent functions entering the point force solution can be obtained from a single function that enters the solution for displacements due to a fluid mass source.

Cleary[1] has established the three dimensional fundamental solutions for a linear elastic fluid-infiltrated porous solid and has outlined their use in modeling embedded regions of inelasticity. However, because of a minor algebraic error in the expressions given by Cleary[1] for the stress field due to point injection of fluid mass at a constant rate, the full extent of the correspondence between point force and fluid mass source solutions is not revealed. This note corrects the expressions given by Cleary[1] and in so-doing fully exposes this correspondence.

To obtain the solutions for fluid mass sources, Cleary[1] uses the procedure of Rice and Cleary[2] for spherically symmetric problems. For injection of fluid mass at the origin at a constant rate q, σ_{rr} and $\sigma_{\theta\theta}$, the only nonzero stress components in polar coordinates, are as follows:

$$\sigma_{rr} = \frac{-\eta q}{2\pi\rho_0\kappa r}\left[\operatorname{erfc}(\xi/2) + \frac{1}{\sqrt{\pi}}\int_0^{\xi}(\eta/\xi)^2 e^{-\eta^2/4}\,d\eta\right] \tag{1a}$$

$$\sigma_{\theta\theta} = \frac{-\eta q}{2\pi\rho_0\kappa r}\left[\frac{1}{2}\operatorname{erfc}(\xi/2) - \frac{1}{2\sqrt{\pi}}\int_0^{\xi}(\eta/\xi)^2 e^{-\eta^2/4}\,d\eta\right]. \tag{1b}$$

The notation follows that used by Cleary[1] (also [2]): r is the radial coordinate; ρ_0 is the fluid mass density; κ is a permeability; $\xi = r/(ct)^{1/2}$, where t is time and c is the diffusivity; and

$$\operatorname{erfc}(z) = (2/\sqrt{\pi})\int_z^{\infty} e^{-x^2}\,dx$$

is the complementary error function. The dimensionless constant η can be expressed as

$$\eta = 3(\nu_u - \nu)/[2B(1+\nu_u)(1-\nu)]$$

where ν and ν_u are the values of Poisson's ratio for drained (long-time) and undrained (short-time) response, respectively and B is the magnitude of the ratio of pore pressure change to mean normal stress during undrained conditions. It will also be convenient to use the following expression for the diffusivity c:

$$c = \mu\kappa(\nu_u - \nu)/[2\eta^2(1-\nu)(1-\nu_u)] \tag{2}$$

where μ is the shear modulus. The stress (1) is expressed in Cartesian components as

$$\sigma_{ij} = \frac{-\eta q}{4\pi\rho_0\kappa}\left\{\frac{\delta_{ij}}{r}[\operatorname{erfc}(\xi/2) - 2\xi^{-2}h(\xi)] + \frac{x_i x_j}{r^3}[\operatorname{erfc}(\xi/2) + 6\xi^{-2}h(\xi)]\right\} \tag{3}$$

where δ_{ij} is the Kronecker delta and

$$h(\xi)=\frac{1}{2\sqrt{\pi}}\int_0^{\xi}\eta^2\,e^{-\eta^2/4}\,d\eta. \tag{4}$$

Equation (1a) agrees with the first of eqns (39a) in [1] but in Cleary's expression for the "hoop stress" $\sigma_{\theta\theta}$, the factor of one-half multiplying erfc $(\xi/2)$ is omitted. Consequently, the Cartesian component form given by (39b) in [1] is incorrect and, because this expression is used to obtain the stress field of a fluid mass dipole, the dipole stress field given in Cleary's equation (43) is also incorrect. For completeness, the expressions for the alteration of pore fluid pressure p (given correctly by equation (38) in [1]) and the displacement components are recorded below:

$$p(\mathbf{x},t)=\frac{q}{\rho_0\kappa}\frac{1}{4\pi r}\operatorname{erfc}(\xi/2) \tag{5}$$

$$u_i(\mathbf{x},t)=\frac{q\eta}{8\pi\rho_0\kappa\mu}\frac{x_i}{r}u(\xi) \tag{6}$$

where

$$u(\xi)=\operatorname{erfc}(\xi/2)+2\xi^{-2}h(\xi). \tag{7}$$

The solution for fluid mass dipoles can be obtained from the source solutions by the usual technique: if a particular field quantity is given by $qF(\mathbf{x},t)$ for a source of strength q, then the corresponding quantity for a dipole of strength q with axis in the direction $\hat{h}_\alpha$ is

$$F_{\mathrm{dipole}}(\mathbf{x},t)=-qh_\alpha\frac{\partial}{\partial x_\alpha}F(\mathbf{x},t). \tag{8}$$

Using q_α to denote qh_α and applying the operation (8) to (5), (6) and (3) yield the following expressions:

$$p(\mathbf{x},t)=\frac{q_\alpha}{4\pi\rho_0\kappa}\frac{x_\alpha}{r^3}[1-h(\xi)], \tag{9}$$

$$u_i(\mathbf{x},t)=\frac{-q_\alpha\eta}{8\pi\rho_0\kappa\mu}\left\{\frac{\delta_{i\alpha}}{r}u(\xi)+\frac{x_ix_\alpha}{r^3}[\xi u'(\xi)-u(\xi)]\right\}, \tag{10}$$

and

$$\sigma_{ij}(\mathbf{x},t)=\frac{q_\alpha\eta}{\rho_o\kappa}\frac{1}{4\pi r^3}\Big\{x_\alpha\delta_{ij}(\xi\Sigma_1'-\Sigma_1)+(\delta_{i\alpha}x_j+\delta_{j\alpha}x_i)\Sigma_2(\xi)$$
$$-3\frac{x_ix_jx_\alpha}{r^2}\left(\Sigma_2-\frac{1}{3}\xi\Sigma_2'\right)\Big\} \tag{11}$$

where

$$\Sigma_1(\xi)=\operatorname{erfc}(\xi/2)-2\xi^{-2}h(\xi)$$
$$\Sigma_2(\xi)=\operatorname{erfc}(\xi/2)+6\xi^{-2}h(\xi).$$

Alternatively, the functions Σ_1 and Σ_2 can be expressed as follows in terms of $u(\xi)$:

$$\Sigma_1(\xi)=u(\xi)+\xi u'(\xi)$$
$$\Sigma_2(\xi)=u(\xi)-\xi u'(\xi).$$

The expression for the stress field (11) corrects eqn (43) in [1]. It is noteworthy that this solution is identical to that for a suddenly applied point force, with components $P_\alpha{}^F = -q_\alpha/\rho_0\kappa$, which acts only on the fluid phase (that is, a body force per unit volume of fluid $P_\alpha{}^F\delta(\mathbf{x})H(t)$ which acts only on the fluid phase; $\delta(\mathbf{x})$ is the three dimensional Dirac delta function and $H(t)$ is the unit step function).

As noted by Cleary[1], the pore fluid pressure $p(\mathbf{x}, t)$ in (9) minus its long time value (as $t \to \infty$, $\xi \to 0$ and $h(0) = 0$) is the pore pressure due to a point force with components given by

$$P_\alpha = q_\alpha\mu/(\rho_0\eta c). \tag{12}$$

(The point force can be regarded as a body force $P_\alpha\delta(\mathbf{x})H(t)$ per unit volume of porous solid, including fluid and solid phases.) The dipole stress and displacement field bear a similar relationship to the point force solution, although, because of the error in the dipole stress field given by Cleary[1], this relationship is not apparent in his paper. In particular, the displacement and stress fields due to sudden application of a point force P_α at the origin can be written as follows:

$$u_i(\mathbf{x}, t) = \frac{P_j}{16\pi r\mu}\frac{1}{(1-\nu_u)}\left\{(3-4\nu_u)\delta_{ij} + \frac{x_ix_j}{r^2}\right\} + \frac{P_j}{16\pi r\mu}\left[\frac{(\nu_u - \nu)}{(1-\nu_u)(1-\nu)}\right]$$
$$\times\left\{u(\xi)\delta_{ij} + \frac{x_ix_j}{r^2}[\xi u'(\xi) - u(\xi)]\right\} \tag{13}$$

$$\sigma_{ij}(\mathbf{x}, t) = \frac{P_k}{8\pi r^3}\frac{1}{(1-\nu_u)}\left\{(1-2\nu_u)[x_k\delta_{ij} - (\delta_{ik}x_j + \delta_{jk}x_i)] - \frac{3x_ix_jx_k}{r^2}\right\}$$
$$-\frac{P_k}{8\pi r^3}\left[\frac{(\nu_u - \nu)}{(1-\nu_u)(1-\nu)}\right]\left\{x_k\delta_{ij}(\xi\Sigma_1' - \Sigma_1) + (\delta_{ik}x_j + \delta_{jk}x_i)\Sigma_2\right.$$
$$\left.-\frac{3x_ix_jx_k}{r^2}\left(\Sigma_2 - \frac{1}{3}\xi\Sigma_2'\right)\right\}. \tag{14}$$

In each of (13) and (14) the first term is the time-independent classical elasticity solution based on the undrained (short-time) elastic constants. This term gives the instantaneous response of the porous medium to the suddenly applied point force. Noting (2) and comparing (10) and (11) with (13) and (14) reveal that the second term in each of the latter equations is the response to a continuous fluid mass dipole with components given by solving (12) for q_α. A corollary of this result is that the time dependence of the point force solution can be derived from the single function $u(\xi)$ (7) which arises in the solution for displacements due to a continuous fluid mass source. As noted by Cleary, the dipole solution contributes no net force on any contour surrounding the origin, and, as a consequence, the point force is equilibrated solely by the first term of (14).

The arrangement of the point force solution in (13) and (14) may prove advantageous in the construction of solutions to boundary value problems and in simulation of zones of inelasticity, as outlined by Cleary. Specifically, some simplification may result from the separation of the solution into two components, one which is the solution to the classical elasticity equations with undrained moduli and one which arises from a fluid mass source solution. However, the difficulty in solving boundary value problems inevitably enters in the coupling introduced in the boundary conditions (e.g. [2]) and it is unlikely that this complication can be avoided.

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